The Biot-Savart Law

So, we now know that given some **current density**, we can find the resulting **magnetic vector potential** $\mathbf{A}(\bar{r})$:

$$\mathbf{A}(\bar{r}) = \frac{\mu_0}{4\pi} \iiint_V \frac{\mathbf{J}(\bar{r}')}{|\bar{r} - \bar{r}'|} \, dv'$$

and then determine the resulting **magnetic flux density** $\mathbf{B}(\bar{r})$ by taking the **curl**:

$$\mathbf{B}(\bar{r}) = \nabla \times \mathbf{A}(\bar{r})$$

Q: ***Golly***, *can't we somehow* ***combine*** *the curl operation and the magnetic vector potential integral?*

A: Yes! The result is known as the **Biot-Savart Law**.

Combining the two above equations, we get:

$$\mathbf{B}(\bar{r}) = \nabla \times \frac{\mu_0}{4\pi} \iiint_V \frac{\mathbf{J}(\bar{r}')}{|\bar{r} - \bar{r}'|} \, dv'$$

This result is of course **not** very helpful, but we note that we can move the curl operation **into** the integrand:

$$\mathbf{B}(\bar{r}) = \frac{\mu_0}{4\pi} \iiint_V \nabla x \frac{\mathbf{J}(\bar{r}')}{|\bar{r} - \bar{r}'|} \, dv'$$

Note this result **reverses** the process: **first** we perform the curl, and **then** we integrate.

We can do this is because the **integral** is over the **primed** coordinates (i.e.,$\bar{r}'$) that specify the **sources** (current density), while the **curl** take the derivatives of the **unprimed** coordinates (i.e., $\bar{r}$) that describe the **fields** (magnetic flux density).

Q: *Yikes! That curl operation still looks particularly **difficult**. How we perform it?*

A: We take advantage of a know **vector identity**! The curl of vector field $f(\bar{r})\mathbf{G}(\bar{r})$, where $f(\bar{r})$ is any **scalar** field and $\mathbf{G}(\bar{r})$ is any **vector** field, can be evaluated as:

$$\nabla x \left(f(\bar{r}) \mathbf{G}(\bar{r}) \right) = f(\bar{r}) \nabla x \mathbf{G}(\bar{r}) - \mathbf{G}(\bar{r}) x \nabla f(\bar{r})$$

Note the **integrand** of the above equation **is** in the form $\nabla x \left(f(\bar{r}) \mathbf{G}(\bar{r}) \right)$, where:

$$f(\bar{r}) = \frac{1}{|\bar{r} - \bar{r}'|} \qquad \text{and} \qquad \mathbf{G}(\bar{r}) = \mathbf{J}(\bar{r}')$$

Therefore we find:

$$\nabla \times \left(\frac{\mathbf{J}(\bar{r}')}{|\bar{r} - \bar{r}'|} \right) = \frac{1}{|\bar{r} - \bar{r}'|} \nabla \times \mathbf{J}(\bar{r}') - \mathbf{J}(\bar{r}') \times \nabla \left(\frac{1}{|\bar{r} - \bar{r}'|} \right)$$

In the **first** term we take the **curl** of $\mathbf{J}(\bar{r}')$. Note however that this vector field is a **constant** with respect to the **unprimed** coordinates $\bar{r}$. Thus the **derivatives** in the curl will all be equal to **zero**, and we find that:

$$\nabla \times \mathbf{J}(\bar{r}') = 0$$

Likewise, it can be shown that:

$$\nabla \left(\frac{1}{|\bar{r} - \bar{r}'|} \right) = -\frac{\bar{r} - \bar{r}'}{|\bar{r} - \bar{r}'|^3}$$

Using these results, we find:

$$\nabla \times \left(\frac{\mathbf{J}(\bar{r}')}{|\bar{r} - \bar{r}'|} \right) = \frac{\mathbf{J}(\bar{r}') \times (\bar{r} - \bar{r}')}{|\bar{r} - \bar{r}'|^3}$$

and therefore the magnetic flux density is:

$$\mathbf{B}(\bar{r}) = \frac{\mu_0}{4\pi} \iiint_V \frac{\mathbf{J}(\bar{r}') \times (\bar{r} - \bar{r}')}{|\bar{r} - \bar{r}'|^3} \, dv'$$

This is know as the **Biot-Savart Law** !

For a **surface** current $\mathbf{J}_s(\bar{\mathrm{r}})$, the Biot-Savart Law becomes:

$$\mathbf{B}(\bar{\mathrm{r}}) = \frac{\mu_0}{4\pi} \iint_S \frac{\mathbf{J}_s(\bar{\mathrm{r}}') \times (\bar{\mathrm{r}} - \bar{\mathrm{r}}')}{|\bar{\mathrm{r}} - \bar{\mathrm{r}}'|^3} \, ds'$$

and for **line** current I, flowing on contour C, the Biot-Savart Law is:

$$\mathbf{B}(\bar{\mathrm{r}}) = \frac{\mu_0 I}{4\pi} \oint_C \frac{\overline{d\ell'} \times (\bar{\mathrm{r}} - \bar{\mathrm{r}}')}{|\bar{\mathrm{r}} - \bar{\mathrm{r}}'|^3}$$

Note the contour C is **closed**. Do **you** know why?

*This is **dad-gum** outstanding! The Biot-Savart Law allows us to **directly** determine magnetic flux density* $\mathbf{B}(\bar{\mathrm{r}})$*, given some current density* $\mathbf{J}(\bar{\mathrm{r}})$*!*

Note that the Biot-Savart Law is therefore **analogous** to **Coloumb's Law** in Electrostatics (Do you see why?)!